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A Novel Analytical Technique of Estimating Whole Life Insurance Benefits Payable Multiple Times Per Period of Insurance

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Abstract

The deepest form of actuarial estimation problems remains the subject of classical life insurance methodologies. A life insurance contract provides the payment of a defined sum assured contingent upon the death of an insured life. Although in practice, death benefits is payable as soon as death claim is advised and the legal requirement is completed, it is necessary to examine death benefits which are paid at the end of policy anniversary of death, that is on the first policy anniversary of effecting the policy after death. When the frequency of payments of an mthly life insurance benefit scheme is infinite, the resulting life insurance function becomes continuously payable momentarily throughout the year so that the total annual payment is equivalent to 1. This admittedly artificial phenomenon has marked consequences in classical life contingency applications and at the same time important as an estimation of benefits payments made weekly or monthly in life insurance benefit program. Consequently, the approximation in the form most suitable for this purpose will be based on Bernoulli power series. In this paper, the objective is to construct analytical expressions for whole life insurance functions payable at different frequencies where the resulting expression represents an adjustment to the yearly formula. Unless an analytical expression for the survival function at age x is defined, approximation will be required to evaluate this expressions. From the results obtained, we confirm asymptotically that $\lim_{K\to\infty} A_x^{(K)} = \overline{A}_x$.

Keywords: Estimation problems, policy anniversary, death benefits, Bernoulli power series, survival function

Introduction

A difficulty level in evaluating life insurance functions from their respective actuarial present values where a life insurance scheme has been incepted under continuous setting is that the probability of survival function for a life aged x may not have an explicit representation. In addition, the derivative of the governing force of mortality may not even exist everywhere. Undoubtedly, there is a gap between the numerical estimations and analytical derivations of life insurance functions. This accounts for the reason why moderate actuarial estimation is required to address complex issues identified in theory to enable us generate closed form expressions which serve as a reference point in a more complex mortality scenario.

Therefore, the theory of estimation is of crucial significance for life insurers to remain solvent and meet the needs of all parties to the business especially the policyholders and stakeholders. A life insurance is a contractual agreement under which the insurer having received premiums from the insured legally accepts a risk from the insured by agreeing to pay benefit contingent on the occurrence of a specified uncertain future event. In Hoem (1969), Hoem (1988), Christiansen (2008) and Christiansen, (2010), the policies are usually long term contracts where the benefit is defined at inception and are underwritten to cover mortality and longevity risks or have embedded savings structure. A typical life insurance is the whole life insurance scheme where the benefit is paid irrespective of the time of death of the insured (Bowers, Gerber, Hickman, Jones, Nesbitt, 1997); Dickson, Hardy & Waters, 2013).

According to Cox, Ingersoll and Ross (1985), Hacaritz, Kleinow and Macdonald (2024), the projection of future cash flows under a life insurance scheme evolves as a result of the requirements to develop key actuarial assumptions in form of technical bases for pricing and satisfying valuation conditions. Following observations in Steffensen (2000), the actuarial assumptions are developed in respect of future interest rates to discount cash flows to the present. Following Sundt and Teugels (2004), the actuarial bases are derived in accordance with future rates of mortality and future expenses as well as basis set in the policy to target profit.

In the classical insurance domain, a level of safety margins is defined when applying the actuarial basis by setting the interest rate below the market level so that a safety margin is built into the mortality rates. Ramlau-Hansen (1988), Ramlau-Hansen (1990) and Linnemann (1993) argue that although life insurers offer various types of life insurance products, safety margins could differ consistent with the kind of policy underwritten. The inclusion of margins is to ensure that on the average, profit emerges over time.

The benefit payment functionally depends on the time of death of the insured or on his survival at a predetermined term. The actuarial methodologies adopted in modelling the uncertainties within the duration of an insured's future lifetime is to consider the remaining lifetime random variable of such life.

The future lifetime of a life aged x is defined by the continuous random variable T_x and the age at death is represented by $T_x + x$ .The cumulative distribution function of T_x applied in computing probabilities of death at time t is given by

$$F_{T_x}(s) = P(T_x \le s) = {}_s q_x \quad (1)$$

while the complementary function is defined as

$$S_{T_x}(s) = 1 - F_{T_x}(s) = P(T_x > s) = {}_s p_x \quad (2)$$

To calculate probabilities at different ages given that a life survives to that age some years later using T_x; $x \geq 0$, it is assumed that

$$F_{T_x}(s) = P(T_x \leq s) = {}_s q_x = P(T_0 \leq x+s | T_0 > x) \quad (3)$$

for all $x \geq 0$ and T_0 is the future lifetime of a newborn. From the axioms of conditional probability, we have

$$F_{T_x}(s) = P(T_x \leq s) = \frac{P(x < T_0 \leq x+s)}{P(T_0 > x)} = \frac{{}_{x+s}q_0 - {}_x q_0}{{}_x p_0} \quad (4)$$

$${}_s q_x + {}_s p_x = 1 \quad (5)$$

the consistency condition for the survival probability requires that

$${}_{x+s} p_0 = ({}_x p_0)({}_s p_x) \quad (6)$$

Consequently, following observations in Dickson et al. (2013), the survival probability of a new born surviving to age $x+s$ is the product of the survival probability from birth to age x and the survival probability from age x to age $x+s$.

An important aspect of mortality is the force of mortality for (x) defined by

$$\mu_x = \lim_{\Delta \to 0} \frac{{}_\Delta q_x}{\Delta} = \lim_{\Delta \to 0^+} \left(\frac{1 - {}_\Delta p_x}{\Delta} \right) \quad (7)$$

defining the relationship between the integrated hazard function and survival probability defined

by ${}_s p_x = \exp\left(-\int_0^s \mu_{x+s} ds \right)$ (8)

The force of mortality is the instantaneous mortality measure on a life aged x. Within a short interval of time Δ, it is assumed that $\Delta \times \mu_x = {}_\Delta q_x$

The death density function of the future lifetime T_x is obtained as follows.

$$f_{T_x}(s) = \frac{d}{ds}\left({}_s q_x\right) = \frac{d}{ds}\left(-{}_s p_x\right) = \left(\mu_{x+s}\right)\left({}_s p_x\right) \tag{9}$$

Following Dickson et al. (2013), we obtain an important formula that relates the future lifetime distribution function in terms of the survival function and the force of mortality

$$ {}_s q_x = \int_0^s {}_s p_x \mu_{x+s} ds \tag{10}$$

Under a life insurance policy, the payment of the benefit by the insurer and the payment of the premium by the insured can either be in the form of a single amount or a life contingent annuity. Lump sum premiums are paid at the beginning of the policy to guarantee risk coverage. The life contingent single benefits and the life contingent annuities depend on the time of death of the policyholder.

Following Anggraeni, Rahmadani, Utama and Handayani. (2023), the valuation of these types of benefits and annuities is essential for the computation of premiums and examination of policy values. The life contingent single benefit is a function of the time of death that is modelled as a random variable. Its present value depends on the chosen actuarial basis. For different actuarial bases, the distribution of the present value can be derived while its actuarial present value and other moments can equally be obtained.

The present value function of a whole life insurance function is given by $e^{-\delta T_x}$ while its actuarial present value is

$$\overline{A}_x = \int_0^{\Omega-x} e^{-\delta s}\left({}_s p_x\right)\mu_{+s} ds \tag{11}$$

Cash flows could occur during the fraction of a year, as for example monthly or quarterly. Considering a fraction of a year $\frac{1}{n}; \quad n \geq 1$, where n can be 12 or 4 corresponding to months or quarters and defining the curtate future lifetime random variable as

$$K_x^{(n)} = \frac{1}{n}\lfloor nT_x \rfloor \tag{12}$$

where $\lfloor . \rfloor$ is the floor function. In this case, the contingent single benefits can be obtained in discrete time at that fraction of the year where $v = \frac{1}{1+i}$ and ${}_{\frac{k}{n}|\frac{1}{n}}q_x$ is the probability probability that the life aged x survives $\frac{k}{n}$ years and then dies in the next $\frac{1}{n}$ years. The present value function of whole life insurance function is given by $v^{K_x^{(n)}+\frac{1}{n}}$ while its actuarial present value is

$$A_x^{(n)} = \sum_{k=0}^{\Omega=x-1}\left(v^{\frac{k+1}{n}}\right)\left({}_{\frac{k}{n}|\frac{1}{n}}q_x\right) \tag{13}$$

Methodology

When life assurance product is designated as $A_x^{(K)}$, then 1 unit of benefit should be paid $\frac{1}{K}$ of year after insurance period s where s is increased at intervals of $\frac{1}{K}$ within $0 \leq s < \Omega$. As $K \to \infty$, $A_x^{(K)} \to \bar{A}_x$. We then apply Euler–Maclaurin model on the kthly payable whole life insurance benefit $A_x^{(K)}$ to obtain $\bar{A}_x$. We define the following nomenclature consistent with Bowers et al., 1997) as follows

Let $\frac{d}{ds}$ define the differential operator.

Δ_1 be the differencing at interval s of 1

$\Sigma^{(1)} = \Sigma$ be the summation operator at interval of 1 to infinity

$\Sigma^{(K)}$ be the summation operator at interval of $\frac{1}{K}$ that is $\left\{0, \frac{1}{K}, \frac{2}{K}, \frac{3}{K}, ..., \frac{K-1}{K}, 1, \frac{K+1}{K}, ...\right\}$ to infinity

δ is the differencing operator in the interval $\frac{1}{K}$

$$(1+\Delta) = (1+\delta)^K = \exp\left(\frac{d}{ds}\right) \tag{14}$$

Obtaining $\frac{1}{\delta}$ in (14) we obtain

$$K\sum^{(k)} = \frac{1}{\delta} = \left[e^{\frac{1}{K}\frac{d}{ds}} - 1\right]^{-1} \Rightarrow \sum^{(k)} = \frac{1}{K\delta} = \frac{1}{K}\left(\frac{1}{\delta}\right) = \frac{1}{K}\left[e^{\frac{1}{K}\frac{d}{ds}} - 1\right]^{-1} \tag{15}$$

Observe that

$$\frac{\frac{1}{K}\frac{d}{ds}}{\left(e^{\frac{1}{K}\frac{d}{ds}} - 1\right)} = \sum_{n=0}^{\infty} B_n \times \left\{\frac{\left(\frac{1}{K}\right)\frac{d^n}{ds^n}}{n!}\right\} \tag{16}$$

where B_n are the Bernoulli numbers

$$\frac{\frac{1}{K}\frac{d}{ds}}{\left(e^{\frac{1}{K}\frac{d}{ds}} - 1\right)} = \frac{B_0}{0!}\left(\frac{d}{Kds}\right)^0 + \frac{B_1}{1!}\left(\frac{1}{K}\frac{d^1}{ds^1}\right) + \frac{B_2}{2!}\left(\frac{1}{K}\frac{d^2}{ds^2}\right) + \frac{B_3}{3!}\left(\frac{1}{K}\frac{d^3}{ds^3}\right) + \frac{B_4}{4!}\left(\frac{1}{K}\frac{d^4}{ds^4}\right)$$
$$+ \frac{B_5}{5!}\left(\frac{1}{K}\frac{d^5}{ds^5}\right) + \frac{B_6}{6!}\left(\frac{1}{K}\frac{d^6}{ds^6}\right) + ... + \frac{B_r}{r!}\left(\frac{1}{K}\frac{d^r}{ds^r}\right) \tag{17}$$

Let

$$y = \frac{1}{K}\frac{d}{ds} \tag{18}$$

$$B_r = \left[\frac{d^m}{dy^m} \left(\frac{y}{e^y - 1} \right) \right]_{y=0} \tag{19}$$

$$B_0 = \lim_{y \to 0} \left(\frac{y}{e^y - 1} \right) = \lim_{y \to 0} \left(\frac{1}{e^y} \right) = 1 \tag{20}$$

$$B_1 = \lim_{y \to 0} \frac{d}{dy} \left(\frac{y}{e^y - 1} \right) = \lim_{y \to 0} \left(\frac{e^y - 1 - ye^y}{\left(e^y - 1\right)^2} \right) = \lim_{y \to 0} \left(\frac{e^y - ye^y - e^y}{2\left(e^y - 1\right)} \right) = \lim_{y \to 0} \left(\frac{-ye^y}{\left(2e^y - 2\right)} \right)$$
$$= \lim_{y \to 0} \left(\frac{-ye^y - e^y}{2e^y} \right) = \lim_{y \to 0} \left(\frac{-y - 1}{2} \right) = \frac{-1}{2} \tag{21}$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} \dots \tag{22}$$

Subtracting 1 from both sides (22)

$$e^y - 1 = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} \dots \tag{23}$$

Dividing both sides by y , we obtain

$$\frac{e^y - 1}{y} = 1 + \frac{y}{2!} + \frac{y^2}{3!} + \frac{y^3}{4!} + \frac{y^4}{5!} \dots \tag{24}$$

$$\text{Let } U = \frac{y}{2!} + \frac{y^2}{3!} + \frac{y^3}{4!} + \frac{y^4}{5!} + \dots \tag{25}$$

$$\frac{e^y - 1}{y} = 1 + U \tag{26}$$

Now observe that the half-angle hyperbolic cotangent function is given by

$$\frac{y}{2} \coth \frac{y}{2} = \frac{y}{2} \frac{e^{\frac{y}{2}} + e^{-\frac{y}{2}}}{e^{\frac{y}{2}} - e^{-\frac{y}{2}}} = \frac{y}{2} \frac{e^{-\frac{y}{2}}}{e^{-\frac{y}{2}}} \frac{e^y + 1}{e^y - 1} = \frac{y}{2} \left(\frac{e^y - 1}{e^y - 1} + \frac{2}{e^y - 1} \right) = \frac{y}{2} + \frac{y}{2} \frac{2}{e^y - 1} = \frac{y}{2} + \frac{y}{e^y - 1} \tag{27}$$

This is the reciprocal of both sides of (24) and is defined in terms of hyperbolic cotangent.

hence

$$\frac{y}{2} \coth \frac{y}{2} - \frac{y}{2} = \frac{y}{e^y - 1} \tag{28}$$

By definition,

$$\frac{y}{2}\coth\frac{y}{2}=\frac{y}{2}\frac{\cosh\frac{y}{2}}{\sinh\frac{y}{2}}=\frac{\frac{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}{2}}{\frac{e^{\frac{y}{2}}-e^{-\frac{y}{2}}}{2}}=\frac{e^{\frac{y}{2}}+e^{-\frac{y}{2}}}{e^{\frac{y}{2}}-e^{-\frac{y}{2}}} \tag{29}$$

Expanding the bottom bracket and simplify

$$\frac{y}{2}\coth\frac{y}{2}=$$
$$\left(\frac{y}{2}\right)\left(\frac{\begin{array}{c}1+\frac{1}{1!}\left(\frac{y}{2}\right)^1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{5!}\left(\frac{y}{2}\right)^5+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\ldots\\ +1+\frac{1}{1!}\left(-\frac{y}{2}\right)^1+\frac{1}{2!}\left(-\frac{y}{2}\right)^2+\frac{1}{3!}\left(-\frac{y}{2}\right)^3+\frac{1}{4!}\left(-\frac{y}{2}\right)^4+\frac{1}{5!}\left(-\frac{y}{2}\right)^5+\left(-\frac{y}{2}\right)^6+\ldots\end{array}}{\begin{array}{c}1+\frac{1}{1!}\left(\frac{y}{2}\right)^1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{5!}\left(\frac{y}{2}\right)^5+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\ldots+\\ -\left(1+\frac{1}{1!}\left(-\frac{y}{2}\right)^1+\frac{1}{2!}\left(-\frac{y}{2}\right)^2+\frac{1}{3!}\left(-\frac{y}{2}\right)^3+\frac{1}{4!}\left(-\frac{y}{2}\right)^4+\frac{1}{5!}\left(-\frac{y}{2}\right)^5+\frac{1}{6!}\left(-\frac{y}{2}\right)^6+\ldots\right)\end{array}}\right) \tag{30}$$

$$\frac{y}{2}\coth\frac{y}{2}=$$
$$\left(\frac{y}{2}\right)\left(\frac{1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\ldots+1+\frac{1}{2!}\left(-\frac{y}{2}\right)^2+\frac{1}{4!}\left(-\frac{y}{2}\right)^4+\left(-\frac{y}{2}\right)^6+\ldots}{\begin{array}{c}1+\frac{1}{1!}\left(\frac{y}{2}\right)^1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{5!}\left(\frac{y}{2}\right)^5+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\ldots+\\ -1-\frac{1}{1!}\left(-\frac{y}{2}\right)^1-\frac{1}{2!}\left(-\frac{y}{2}\right)^2-\frac{1}{3!}\left(-\frac{y}{2}\right)^3-\frac{1}{4!}\left(-\frac{y}{2}\right)^4-\frac{1}{5!}\left(-\frac{y}{2}\right)^5-\frac{1}{6!}\left(-\frac{y}{2}\right)^6-\ldots\end{array}}\right) \tag{31}$$

$$\frac{y}{2}\coth\frac{y}{2}=$$
$$\left(\frac{y}{2}\right)\left(\frac{1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{6!}\left(\frac{y}{2}\right)^6+...+1+\frac{1}{2!}\left(-\frac{y}{2}\right)^2+\frac{1}{4!}\left(-\frac{y}{2}\right)^4+\left(-\frac{y}{2}\right)^6+...}{\frac{1}{1!}\left(\frac{y}{2}\right)^1+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{5!}\left(\frac{y}{2}\right)^5+...+\frac{1}{1!}\left(\frac{y}{2}\right)^1+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{5!}\left(\frac{y}{2}\right)^5-...}\right) \quad (32)$$

$$\frac{y}{2}\coth\frac{y}{2}=\left(\frac{y}{2}\right)\left(\frac{2+2\times\frac{1}{2!}\left(\frac{y}{2}\right)^2+2\times\frac{1}{4!}\left(\frac{y}{2}\right)^4+2\times\frac{1}{6!}\left(\frac{y}{2}\right)^6+...}{2\times\frac{1}{1!}\left(\frac{y}{2}\right)^1+2\times\frac{1}{3!}\left(\frac{y}{2}\right)^3+2\times\frac{1}{5!}\left(\frac{y}{2}\right)^5+...}\right) \quad (33)$$

Dividing the numerator and denominator of the right hand side by 2 simplifies to

$$\frac{y}{2}\coth\frac{y}{2}=\left(\frac{y}{2}\right)\left(\frac{1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\frac{1}{8!}\left(\frac{y}{2}\right)^8+\frac{1}{10!}\left(\frac{y}{2}\right)^{10}+...}{\left(\frac{y}{2}\right)+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{5!}\left(\frac{y}{2}\right)^5+\frac{1}{7!}\left(\frac{y}{2}\right)^7+\frac{1}{9!}\left(\frac{y}{2}\right)^9+\frac{1}{11!}\left(\frac{y}{2}\right)^{11}+...}\right) \quad (34)$$

$$\frac{y}{2}\coth\frac{y}{2}=\left[1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\frac{1}{8!}\left(\frac{y}{2}\right)^8+\frac{1}{10!}\left(\frac{y}{2}\right)^{10}\right]\times$$
$$\left(\frac{\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)+\frac{1}{3!}\left(\frac{y}{2}\right)^3+\frac{1}{5!}\left(\frac{y}{2}\right)^5+\frac{1}{7!}\left(\frac{y}{2}\right)^7+\frac{1}{9!}\left(\frac{y}{2}\right)^9+\frac{1}{11!}\left(\frac{y}{2}\right)^{11}+...}\right) \quad (35)$$

Dividing the numerator and denominator in (35) by $\frac{y}{2}$, we obtain

$$\frac{y}{2}\coth\frac{y}{2}=\left(1+\frac{1}{2!}\left(\frac{y}{2}\right)^2+\frac{1}{4!}\left(\frac{y}{2}\right)^4+\frac{1}{6!}\left(\frac{y}{2}\right)^6+\frac{1}{8!}\left(\frac{y}{2}\right)^8+\frac{1}{10!}\left(\frac{y}{2}\right)^{10}\right)$$
$$\times\left(\frac{1}{1+\frac{1}{3!}\left(\frac{y}{2}\right)^2+\frac{1}{5!}\left(\frac{y}{2}\right)^4+\frac{1}{7!}\left(\frac{y}{2}\right)^6+\frac{1}{9!}\left(\frac{y}{2}\right)^8+\frac{1}{11!}\left(\frac{y}{2}\right)^{10}+...}\right) \quad (36)$$

$$\frac{1}{1+U}=1+(-U)+(-U)^2+(-U)^3+(-U)^4+(-U)^5+\dots \quad (37)$$

$$\frac{y}{2}\coth\frac{y}{2}=\left(1+\frac{1}{2!}\left[\frac{y}{2}\right]^2+\frac{1}{4!}\left[\frac{y}{2}\right]^4+\frac{1}{6!}\left[\frac{y}{2}\right]^6+\frac{1}{8!}\left[\frac{y}{2}\right]^8+\frac{1}{10!}\left[\frac{y}{2}\right]^{10}\right)$$
$$\times\left\{\begin{matrix}\left[1-\left(\frac{1}{3!}\left[\frac{y}{2}\right]^2+\frac{1}{5!}\left[\frac{y}{2}\right]^4+\frac{1}{7!}\left[\frac{y}{2}\right]^6+\frac{1}{9!}\left[\frac{y}{2}\right]^8+\dots\right)\right] \\ +\left(\frac{1}{3!}\left[\frac{y}{2}\right]^2+\frac{1}{5!}\left[\frac{y}{2}\right]^4+\frac{1}{7!}\left[\frac{y}{2}\right]^6\right)^2-\left(\frac{1}{3!}\left[\frac{y}{2}\right]^2+\frac{1}{5!}\left[\frac{y}{2}\right]^4\right)^3+\left(\frac{1}{3!}\left[\frac{y}{2}\right]^2\right)^4\end{matrix}\right\}+O\left(y^{10}\right) \quad (38)$$

After simplifying this equation, we obtain

$$\frac{y}{2}\coth\frac{y}{2}=1+\left(\frac{1}{6}\right)\frac{y^2}{2!}+\left(-\frac{1}{30}\right)\frac{y^4}{4!}+\left(\frac{1}{42}\right)\frac{y^6}{6!}+\left(-\frac{1}{30}\right)\frac{y^8}{8!}+\dots \quad (39)$$

Observe here that the there is no term containing y^1 in equation (39)

But

$$\frac{y}{\left[e^y-1\right]}=\sum_{m=0}^{\infty}\left(\frac{B_m}{m!}\right)y^m=\left(\frac{B_0}{0!}\right)y^0+\left(\frac{B_1}{1!}\right)y^1+\left(\frac{B_2}{2!}\right)y^2+\left(\frac{B_3}{3!}\right)y^3+\left(\frac{B_4}{4!}\right)y^4+\left(\frac{B_5}{5!}\right)y^5$$
$$+\left(\frac{B_6}{6!}\right)y^6+\left(\frac{B_7}{7!}\right)y^7+\left(\frac{B_8}{8!}\right)y^8+\left(\frac{B_9}{9!}\right)y^9+\dots \quad (40)$$

Since there is no term containing y^1 in equation (39) we must add $\left(-\frac{y}{2}\right)$ from both sides

$$\frac{y}{2}\coth\frac{y}{2}-\left(\frac{1}{2}\right)\frac{y}{1!}=-\left(\frac{1}{2}\right)\frac{y}{1!}+1+\left(\frac{1}{6}\right)\frac{y^2}{2!}+\left(-\frac{1}{30}\right)\frac{y^4}{4!}+\left(\frac{1}{42}\right)\frac{y^6}{6!}+\left(-\frac{1}{30}\right)\frac{y^8}{8!}+\dots \quad (41)$$

Comparing co-efficient of powers of y in equations (39) and (40), we have

$$\frac{B_0}{0!}=1;\frac{B_1}{1!}=-\frac{1}{2\times1}=-\frac{1}{2};\frac{B_2}{2!}=\frac{1}{12};\frac{B_3}{3!}=0;\frac{B_4}{4!}=-\frac{1}{720};\frac{B_5}{5!}=0;\frac{B_6}{6!}=-\frac{1}{30240};$$
$$\frac{B_8}{8!}=\frac{1}{30\times40,320} \quad (42)$$

The odd Bernoulli numbers all vanishes except B_1 that is $B_{2r-1} = 0$ for $r \geq 2$. We use the symbol

$\left(\frac{1}{K}\frac{d}{ds}\right)^n$ to mean $\left(\frac{1}{K}D\right)^n$ where $D^n = \frac{d^n}{ds^n}$

$$\frac{\frac{d}{Kds}}{\left[e^{\frac{1}{K}\frac{d}{ds}} - 1\right]} = \frac{1}{0!}\left(\frac{d}{Kds}\right)^0 - \frac{1}{2}\left(\frac{d}{Kds}\right)^1 + \frac{1}{12}\left(\frac{d}{Kds}\right)^2 - \frac{1}{720}\left(\frac{d}{Kds}\right)^4 + \frac{1}{30240}\left(\frac{d}{Kds}\right)^6 - \frac{1}{1209600}\left(\frac{d}{Kds}\right)^8 + \frac{1}{47900160}\left(\frac{d}{Kds}\right)^{10} \tag{43}$$

Dividing throughout by $\frac{d}{Kds}$

$$\frac{1}{\frac{d}{Kds}}\frac{\frac{d}{Kds}}{\left[e^{\frac{1}{K}\frac{d}{ds}} - 1\right]} = \frac{\frac{1}{0!}\left(\frac{d}{Kds}\right)^0}{\frac{d}{Kds}} - \frac{1}{2} + \frac{1}{12}\left(\frac{d}{Kds}\right)^1 - \frac{1}{720}\left(\frac{d}{Kds}\right)^3 + \frac{1}{30240}\left(\frac{d}{Kds}\right)^5 - \frac{1}{1209600}\left(\frac{d}{Kds}\right)^7 + \frac{1}{47900160}\left(\frac{d}{Kds}\right)^9 \tag{44}$$

$$K\sum^{(K)} = \frac{1}{\frac{d}{Kds}}\frac{\frac{d}{Kds}}{\left[e^{\frac{1}{K}\frac{d}{ds}} - 1\right]} \tag{45}$$

$$K\sum^{(K)} = \frac{1}{\frac{d}{Kds}} - \frac{1}{2} + \frac{1}{12}\left(\frac{d}{Kds}\right)^1 - \frac{1}{720}\left(\frac{d}{Kds}\right)^3 + \frac{1}{30240}\left(\frac{d}{Kds}\right)^5 - \frac{1}{1209600}\left(\frac{d}{Kds}\right)^7 + \frac{1}{47900160}\left(\frac{d}{Kds}\right)^9 \tag{46}$$

$$\sum^{(K)} = \frac{1}{\left(\frac{K}{1}\times\frac{d}{Kds}\right)} - \frac{1}{2K} + \frac{1}{12K}\left(\frac{d}{Kds}\right)^{1} - \frac{1}{720K}\left(\frac{d}{Kds}\right)^{3} + \frac{1}{30240K}\left(\frac{d}{Kds}\right)^{5} - \frac{1}{1209600K}\left(\frac{d}{Kds}\right)^{7} + \frac{1}{47900160K}\left(\frac{d}{Kds}\right)^{9} \quad (47)$$

Note that $\left(\frac{d}{ds}\right)^{n} = D^{n} = \frac{d^{n}}{ds^{n}}$ and substituting $K=1$ in (47)

$$\sum^{(1)} = \frac{1}{\left(\frac{d}{ds}\right)} - \frac{1}{2} + \frac{1}{12}\left(\frac{d}{ds}\right)^{1} - \frac{1}{720}\left(\frac{d}{ds}\right)^{3} + \frac{1}{30240}\left(\frac{d}{ds}\right)^{5} - \frac{1}{1209600}\left(\frac{d}{ds}\right)^{7} + \frac{1}{47900160}\left(\frac{d}{ds}\right)^{9} \quad (48)$$

$$\sum^{(K)} - \sum^{(1)} = \frac{1}{K\frac{d}{Kds}} - \frac{1}{2K} + \frac{1}{12K^{2}}\left(\frac{d}{ds}\right)^{1} - \frac{1}{720K^{4}}\left(\frac{d}{ds}\right)^{3} + \frac{1}{30240K^{6}}\left(\frac{d}{ds}\right)^{5} - \frac{1}{1209600K^{8}}\left(\frac{d}{ds}\right)^{7} + \frac{1}{47900160K^{10}}\left(\frac{d}{ds}\right)^{9} - \frac{1}{\left(\frac{d}{ds}\right)} + \frac{1}{2} - \frac{1}{12}\left(\frac{d}{ds}\right)^{1} + \frac{1}{720}\left(\frac{d}{ds}\right)^{3} - \frac{1}{30240}\left(\frac{d}{ds}\right)^{5} + \frac{1}{1209600}\left(\frac{d}{ds}\right)^{7} - \frac{1}{47900160}\left(\frac{d}{ds}\right)^{9} \quad (49)$$

$$\sum^{(K)} - \sum^{(1)} = \frac{1}{K\frac{d}{Kds}} - \frac{1}{\left(\frac{d}{ds}\right)} + \frac{1}{2} - \frac{1}{2K} + \frac{1}{12K^{2}}\left(\frac{d}{ds}\right)^{1} - \frac{1}{12}\left(\frac{d}{ds}\right)^{1} + \frac{1}{720}\left(\frac{d}{ds}\right)^{3} - \frac{1}{720K^{4}}\left(\frac{d}{ds}\right)^{3} + \frac{1}{30240K^{6}}\left(\frac{d}{ds}\right)^{5} - \frac{1}{30240}\left(\frac{d}{ds}\right)^{5} + \frac{1}{1209600}\left(\frac{d}{ds}\right)^{7} - \frac{1}{1209600K^{8}}\left(\frac{d}{ds}\right)^{7} + \frac{1}{47900160K^{10}}\left(\frac{d}{ds}\right)^{9} - \frac{1}{47900160}\left(\frac{d}{ds}\right)^{9} \quad (50)$$

$$\sum^{(K)}-\sum^{(1)}=\frac{K-1}{2K}+\frac{1-K^2}{12K^2}\left(\frac{d}{ds}\right)^1+\frac{K^4-1}{720K^4}\left(\frac{d}{ds}\right)^3+\frac{1-K^6}{30240K^6}\left(\frac{d}{ds}\right)^5 +\frac{K^8-1}{1209600K^8}\left(\frac{d}{ds}\right)^7+\frac{1-K^{10}}{47900160K^{10}}\left(\frac{d}{ds}\right)^9 \tag{51}$$

We insert the function $\frac{C_{x+s}}{D_x}$ in (51) throughout as follows

$$\sum^{(K)}\left(\frac{C_{x+s}}{D_x}\right)-\sum^{(1)}\left(\frac{C_{x+s}}{D_x}\right)=\frac{K-1}{2K}\left(\frac{C_{x+s}}{D_x}\right)+\frac{1-K^2}{12K^2}\left(\frac{d}{ds}\frac{C_{x+s}}{D_x}\right)^1+\frac{K^4-1}{720K^4}\left(\frac{d}{ds}\frac{C_{x+s}}{D_x}\right)^3 +\frac{1-K^6}{30240K^6}\left(\frac{d}{ds}\frac{C_{x+s}}{D_x}\right)^5+\frac{K^8-1}{1209600K^8}\left(\frac{d}{ds}\frac{C_{x+s}}{D_x}\right)^7+\frac{1-K^{10}}{47900160K^{10}}\left(\frac{d}{ds}\frac{C_{x+s}}{D_x}\right)^9 \tag{52}$$

Since the integral $\bar{A}_x=\int_0^\infty e^{-\delta s}\mu_{x+s}\left({}_sP_x\right)ds$ is difficult to solve by direct integration, we first obtain the derivatives of the *discounted death* function C_{x+s} in equation (52) and thereafter estimate the whole life insurance $\bar{A}_x$. Both the discounted deaths and number of deaths are defined as follows

$$C_x=d_xv^{x+1};\quad d_x=l_x-l_{x+1} \tag{53}$$

where $C_x\in\mathbf{C}^n$ and C_x is the discounted deaths. Replacing x by $x+s$ in (53)

$$C_{x+s}=v^{x+1+s}d_{x+s}=e^{(\ln v)(x+1+s)}d_{x+s} \tag{54}$$

The number of death cases at age $x+s$ is given by

$$d_{x+s}=\left(l_{x+s}-l_{x+1+s}\right) \tag{55}$$

$$C_{x+s}=v^{x+1+s}d_{x+s}=\exp\left[(\ln v)(x+1+s)\right]d_{x+s}=\left(l_{x+s}-l_{x+1+s}\right)\times\exp\left[(\ln v)(x+1+s)\right]= \left(l_{x+s}\exp\left[(\ln v)\times(x+1+s)\right]-l_{x+1+s}\exp\left[(\ln v)\times(x+1+s)\right]\right) \tag{56}$$

Finding the difference between omega age $\Omega=\infty$ and age 0, we have

$$\left[C_{x+s}\right]_0^{\Omega}=\left[C_{x+s}\right]_{s=\Omega}-\left[C_{x+s}\right]_{s=0}=-e^{(\ln v)(x+1)}d_x=-e^{(-\delta)(x+1)}d_x=-v^{x+1}d_x=-C_x \tag{57}$$

where Ω is the limit of life

$$\Omega=Sup\left\{\zeta\in\mathbf{R}^+\middle|F_{T_x}(\zeta)\le 1\right\} \tag{58}$$

Differentiating (56) with respect to s once and observing that

$$\frac{d}{ds}l_{x+s}=-\mu_{x+s}l_{x+s} \tag{59}$$

$$\begin{aligned}\frac{d}{ds}C_{x+s}&=\frac{d}{ds}\left[\left(l_{x+s}-l_{x+1+s}\right)\times\exp\left\{(\ln v)(x+1+s)\right\}\right]=\\&+\left(l_{x+s}-l_{x+1+s}\right)\frac{d}{ds}\exp\left[(\ln v)(x+1+s)\right]+\exp\left[(\ln v)(x+1+s)\right]\frac{d}{ds}\left(l_{x+s}-l_{x+1+s}\right)\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{d}{ds}C_{x+s}&=\left(l_{x+s}-l_{x+1+s}\right)(\ln v)\exp\left[(\ln v)(x+1+s)\right]\\&-\exp\left[(\ln v)(x+1+s)\right]\left[\mu_{x+s}l_{x+s}-\mu_{x+1+s}l_{x+1+s}\right]\end{aligned} \tag{61}$$

$$\begin{aligned}\frac{d}{ds}C_{x+s}&=d_{x+s}(\ln v)\exp\left[(\ln v)(x+1+s)\right]\\&-\exp\left[(\ln v)(x+1+s)\right]\left[\mu_{x+s}l_{x+s}-\mu_{x+1+s}l_{x+1+s}\right]\end{aligned} \tag{62}$$

$$\begin{aligned}\frac{d}{ds}C_{x+s}&=(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+s}l_{x+s}\right)\\&+\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+1+s}l_{x+1+s}\right)\end{aligned} \tag{63}$$

$$\begin{aligned}\left[\frac{d}{ds}C_{x+s}\right]_0^{\Omega}&=\begin{bmatrix}(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+s}l_{x+s}\right)\\+\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+1+s}l_{x+1+s}\right)\end{bmatrix}_{s=\Omega}\\&-\begin{bmatrix}(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+s}l_{x+s}\right)\\+\exp\left[(\ln v)\times(x+1+s)\right]\left(\mu_{x+1+s}l_{x+1+s}\right)\end{bmatrix}_{s=0}\end{aligned} \tag{64}$$

$$\left[\frac{d}{ds}C_{x+s}\right]_0^{\Omega}=-\begin{bmatrix}(\ln v)\exp\left[(\ln v)\times(x+1)\right]\times d_x-\exp\left[(\ln v)\times(x+1)\right]\left(\mu_x l_x\right)\\+\exp\left[(\ln v)\times(x+1)\right]\left(\mu_{x+1}l_{x+1}\right)\end{bmatrix} \tag{65}$$

Taking the second derivative using (63) and noting that

$$\begin{aligned}&\frac{d^2}{ds^2}C_{x+s}=(\ln v)\frac{d}{ds}C_{x+s}+\frac{d}{ds}\left[\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)\right]\\&-\frac{d}{ds}\left[\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)\right]\end{aligned}\tag{66}$$

$$\begin{aligned}&\frac{d^2}{ds^2}C_{x+s}=(\ln v)\begin{bmatrix}(\ln v)C_{x+s}+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)\\-\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)\end{bmatrix}+\\&\left(\frac{d}{ds}l_{x+s}\right)\frac{d}{ds}\exp\left[(\ln v)\times(x+1+s)\right]+\exp\left[(\ln v)\times(x+1+s)\right]\frac{d}{ds}\left(\frac{d}{ds}l_{x+s}\right)\\&-\left(\frac{d}{ds}l_{x+1+s}\right)\frac{d}{ds}\exp\left[(\ln v)\times(x+1+s)\right]-\exp\left[(\ln v)\times(x+1+s)\right]\frac{d}{ds}\left(\frac{d}{ds}l_{x+1+s}\right)\end{aligned}\tag{67}$$

$$\begin{aligned}&\frac{d^2}{ds^2}C_{x+s}=(\ln v)\begin{bmatrix}(\ln v)C_{x+s}+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)\\-\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)\end{bmatrix}\\&+(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}l_{x+s}\right)\\&-(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)-\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}l_{x+1+s}\right)\end{aligned}\tag{68}$$

$$\begin{aligned}&\frac{d^2}{ds^2}C_{x+s}=(\ln v)^2C_{x+s}+2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)\\&-2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)\\&+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}l_{x+s}\right)-\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}l_{x+1+s}\right)\end{aligned}\tag{69}$$

$$\begin{aligned}&\frac{d^2}{ds^2}C_{x+s}=(\ln v)^2C_{x+s}+2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+s}\right)\\&-2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}l_{x+1+s}\right)+\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}\left(l_{x+s}-l_{x+1+s}\right)\end{aligned}\tag{70}$$

$$\frac{d^2}{ds^2}C_{x+s} = (\ln v)^2 C_{x+s} + 2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\frac{d}{ds}\left(l_{x+s} - l_{x+1+s}\right) + \exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}\left(l_{x+s} - l_{x+1+s}\right) \tag{71}$$

$$\frac{d^2}{ds^2}C_{x+s} = (\ln v)^2 C_{x+s} + 2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\frac{d}{ds}d_{x+s} + \exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}d_{x+s} \tag{72}$$

$$\left[\frac{d^2}{ds^2}C_{x+s}\right]_0^{\Omega} = \left[\begin{matrix}(\ln v)^2 C_{x+s} + 2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}d_{x+s}\right) \\ + \exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}d_{x+s}\end{matrix}\right]_{s=\Omega} - \left[\begin{matrix}(\ln v)^2 C_{x+s} + 2(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}d_{x+s}\right) \\ + \exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}d_{x+s}\end{matrix}\right]_{s=0} \tag{73}$$

$$\left[\frac{d^2}{ds^2}C_{x+s}\right]_0^{\Omega} = -\left[\begin{matrix}(\ln v)^2 C_x + 2(\ln v)\exp\left[(\ln v)\times(x+1)\right]\left(\frac{d}{dx}d_x\right) \\ + \exp\left[(\ln v)\times(x+1)\right]\frac{d^2}{dx^2}d_x\end{matrix}\right] \tag{74}$$

The first term will be zero because both C and d are all functions of l_{x+s} which also vanishes at

$$s = \Omega \quad ; \left[\frac{d}{ds}d_{x+s}\right]_{s=0} = \frac{d}{dx}d_x \tag{75}$$

Taking the third derivative

$$\frac{d^3}{ds^3}C_{x+s} = (\ln v)^2 \frac{d}{ds}C_{x+s} + 2(\ln v)\frac{d}{ds}\left[\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}d_{x+s}\right)\right] + \frac{d}{ds}\left[\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}d_{x+s}\right] \tag{76}$$

$$\frac{d^3}{ds^3}C_{x+s}=(\ln v)^2\left\{\begin{aligned}&(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+s}l_{x+s})\\&+\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+1+s}l_{x+1+s})\end{aligned}\right\}$$
$$+2(\ln v)\left\{\left(\frac{d}{ds}d_{x+s}\right)\frac{d}{ds}\exp\left[(\ln v)\times(x+1+s)\right]+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}d_{x+s}\right)\right\} \quad (77)$$
$$+\left[\frac{d^2}{ds^2}d_{x+s}\times\frac{d}{ds}\exp\left[(\ln v)\times(x+1+s)\right]+\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^3}{ds^3}d_{x+s}\right]$$

$$\frac{d^3}{ds^3}C_{x+s}=(\ln v)^2\left\{\begin{aligned}&(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+s}l_{x+s})\\&+\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+1+s}l_{x+1+s})\end{aligned}\right\}$$
$$+2(\ln v)\left[\left(\frac{d}{ds}d_{x+s}\right)(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]+\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}d_{x+s}\right)\right] \quad (78)$$
$$+\left[(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^2}{ds^2}d_{x+s}+\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^3}{ds^3}d_{x+s}\right]$$

$$\frac{d^3}{ds^3}C_{x+s}=(\ln v)^2\left\{\begin{aligned}&(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\times d_{x+s}-\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+s}l_{x+s})\\&+\exp\left[(\ln v)\times(x+1+s)\right](\mu_{x+1+s}l_{x+1+s})\end{aligned}\right\}$$
$$+2(\ln v)^2\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d}{ds}d_{x+s}\right)+3(\ln v)\exp\left[(\ln v)\times(x+1+s)\right]\left(\frac{d^2}{ds^2}d_{x+s}\right) \quad (79)$$
$$+\exp\left[(\ln v)\times(x+1+s)\right]\frac{d^3}{ds^3}d_{x+s}$$

$$\left[\frac{d^3}{ds^3}C_{x+s}\right]_0^\Omega = \left[\begin{array}{l} (\ln v)^2 \left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+1+s)} \times d_{x+s} - e^{(\ln v)(x+1+s)}\left(\mu_{x+s} l_{x+s}\right) \\ +e^{(\ln v)(x+1+s)}\left(\mu_{x+1+s} l_{x+1+s}\right)\end{array}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+1+s)}\left(\frac{d}{ds}d_{x+s}\right)+3(\ln v)e^{(\ln v)(x+1+s)}\left(\frac{d^2}{ds^2}d_{x+s}\right) \\ +e^{(\ln v)(x+1+s)}\frac{d^3}{ds^3}d_{x+s} \end{array}\right]_{s=\Omega}$$

$$-\left[\begin{array}{l} (\ln v)^2 \left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+1+s)} \times d_{x+s} - e^{(\ln v)(x+1+s)}\left(\mu_{x+s} l_{x+s}\right) \\ +e^{(\ln v)(x+1+s)}\left(\mu_{x+1+s} l_{x+1+s}\right)\end{array}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+1+s)}\left(\frac{d}{ds}d_{x+s}\right)+3(\ln v)e^{(\ln v)(x+1+s)}\left(\frac{d^2}{ds^2}d_{x+s}\right) \\ +e^{(\ln v)(x+1+s)}\frac{d^3}{ds^3}d_{x+s} \end{array}\right]_{s=0} \quad (80)$$

$$\left[\frac{d^3}{ds^3}C_{x+s}\right]_0^\Omega = -\left\{\begin{array}{l} (\ln v)^2 \left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+1)} \times d_{x} - e^{(\ln v)(x+1)}\left(\mu_{x} l_{x}\right) \\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right)\end{array}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_{x}\right)+3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_{x}\right) \\ +e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_{x} \end{array}\right\} \quad (81)$$

summing up the discounted death from zero to the limit of life Ω, we have

$$\frac{1}{D_x}\sum_{S=0}^{\Omega}{}^{(K)}C_{x+s} = \frac{1}{D_x}\sum_{S=0}^{\Omega}C_{x+s} + \frac{K-1}{2K}\left[\frac{C_{x+s}}{D_x}\right]_{s=0}^{\Omega} + \frac{\left(1-K^2\right)}{12K^2}\left[\frac{1}{D_x}\frac{d}{ds}C_{x+s}\right]_{s=0}^{\Omega} + \frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_x}\frac{d^3}{ds^3}C_{x+s}\right]_{s=0}^{\Omega} \quad (82)$$

substituting for the derivatives in (82) but ignoring higher order derivatives than 3 gives

$$\frac{1}{D_x}\sum_{S=0}^{\Omega}{}^{(K)}C_{x+s}=\frac{1}{D_x}\sum_{S=0}^{\Omega}C_{x+s}-\frac{K-1}{2K}\left[\frac{C_x}{D_x}\right]-\frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_x}\begin{Bmatrix}(\ln v)e^{(\ln v)(x+1)}\times d_x-e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\+e^{(\ln v)(x+1)}\left(\mu_{x+1}l_{x+1}\right)\end{Bmatrix}\right]-\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_x}\begin{Bmatrix}(\ln v)^2\begin{Bmatrix}(\ln v)e^{(\ln v)(x+1)}\times d_x-e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\+e^{(\ln v)(x+1)}\left(\mu_{x+1}l_{x+1}\right)\end{Bmatrix}\\+2(\ln v)^2e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_x\right)+3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_x\right)\\+e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_x\end{Bmatrix}\right] \tag{83}$$

Following definition in Bowers et al. (1997), the discrete whole life insurance whose benefit is payable at the end of the next anniversary period is defined as

$$A_x=\frac{1}{D_x}\sum_{s=0}^{\Omega}C_{x+s} \tag{84}$$

Consequently, the K-thly life insurance benefit is given as

$$A_x^{(K)}=\frac{1}{D_x}\sum_{S=0}^{\Omega}{}^{(K)}C_{x+s} \tag{85}$$

$$A_x^{(K)}=A_x-\frac{K-1}{2K}\left[\frac{C_x}{D_x}\right]-\frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_x}\begin{Bmatrix}(\ln v)e^{(\ln v)(x+1)}\times d_x-e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\+e^{(\ln v)(x+1)}\left(\mu_{x+1}l_{x+1}\right)\end{Bmatrix}\right]-\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_x}\begin{Bmatrix}(\ln v)^2\begin{Bmatrix}(\ln v)e^{(\ln v)(x+1)}\times d_x-e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\+e^{(\ln v)(x+1)}\left(\mu_{x+1}l_{x+1}\right)\end{Bmatrix}\\+2(\ln v)^2e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_x\right)+3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_x\right)\\+e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_x\end{Bmatrix}\right] \tag{86}$$

$$A_{x+1}{}^{(K)} = A_{x+1} - \frac{K-1}{2K}\left[\frac{C_{x+1}}{D_{x+1}}\right] - \frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_{x+1}}\left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+2)} \times d_{x+1} - e^{(\ln v)(x+2)}\left(\mu_{x+1}l_{x+1}\right) \\ +e^{(\ln v)(x+2)}\left(\mu_{x+2}l_{x+2}\right)\end{array}\right\}\right]$$
$$-\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_{x+1}}\left\{\begin{array}{l}(\ln v)^2\left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+2)} \times d_{x+1} - e^{(\ln v)(x+2)}\left(\mu_{x+1}l_{x+1}\right) \\ +e^{(\ln v)(x+2)}\left(\mu_{x+2}l_{x+2}\right)\end{array}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+2)}\left(\frac{d}{dx}d_{x+1}\right) + 3(\ln v)e^{(\ln v)(x+2)}\left(\frac{d^2}{dx^2}d_{x+1}\right) \\ +e^{(\ln v)(x+2)}\frac{d^3}{dx^3}d_{x+1}\end{array}\right\}\right] \quad (87)$$

$$A_{x+2}{}^{(K)} = A_{x+2} - \frac{K-1}{2K}\left[\frac{C_{x+2}}{D_{x+2}}\right] - \frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_{x+2}}\left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+3)} \times d_{x+2} - e^{(\ln v)(x+3)}\left(\mu_{x+2}l_{x+2}\right) \\ +e^{(\ln v)(x+3)}\left(\mu_{x+3}l_{x+3}\right)\end{array}\right\}\right]$$
$$-\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_{x+2}}\left\{\begin{array}{l}(\ln v)^2\left\{\begin{array}{l}(\ln v)e^{(\ln v)(x+3)} \times d_{x+2} - e^{(\ln v)(x+3)}\left(\mu_{x+2}l_{x+2}\right) \\ +e^{(\ln v)(x+3)}\left(\mu_{x+3}l_{x+3}\right)\end{array}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+3)}\left(\frac{d}{dx}d_{x+2}\right) + 3(\ln v)e^{(\ln v)(x+3)}\left(\frac{d^2}{dx^2}d_{x+2}\right) \\ +e^{(\ln v)(x+3)}\frac{d^3}{dx^3}d_{x+2}\end{array}\right\}\right] \quad (88)$$

Result

As an immediate consequence from the results obtained in equation (86), we can take the limit as K tends to infinity in (86), we obtain the continuous whole life insurance. Following Neil (1979), $\lim_{K\to\infty} A_x^{(K)} = \overline{A}_x$ and consequently,

$$\lim_{K\to\infty} A_x^{(K)} =$$

$$\lim_{K\to\infty} A_x - \lim_{K\to\infty}\frac{K-1}{2K}\left[\frac{C_x}{D_x}\right] - \lim_{K\to\infty}\frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)}\times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right)\end{matrix}\right\}\right]$$

$$-\lim_{K\to\infty}\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)^2\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)}\times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right)\end{matrix}\right\}\\ +2(\ln v)^2 e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_x\right)+3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_x\right)\\ +e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_x\end{matrix}\right\}\right] \quad (89)$$

$$\lim_{K\to\infty} A_x^{(K)} = \lim_{K\to\infty} A_x - \lim_{K\to\infty}\frac{1-\frac{1}{K}}{2}\left[\frac{C_x}{D_x}\right]$$

$$-\lim_{K\to\infty}\frac{\left(1-\frac{1}{K^2}\right)}{12}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)}\times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right)\end{matrix}\right\}\right]$$

$$-\lim_{K\to\infty}\frac{\left(1-\frac{1}{K^4}\right)}{720}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)^2\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)}\times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right)\\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right)\end{matrix}\right\}\\ +2(\ln v)^2 e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_x\right)+3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_x\right)\\ +e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_x\end{matrix}\right\}\right] \quad (90)$$

$$\bar{A}_x = A_x - \frac{1}{2}\left[\frac{C_x}{D_x}\right] - \frac{1}{12}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)} \times d_x - e^{(\ln v)(x+1)}(\mu_x l_x) \\ +e^{(\ln v)(x+1)}(\mu_{x+1} l_{x+1})\end{matrix}\right\}\right]$$

$$- \frac{1}{720}\left[\frac{1}{D_x}\left\{\begin{matrix}(\ln v)^2\left\{\begin{matrix}(\ln v)e^{(\ln v)(x+1)} \times d_x - e^{(\ln v)(x+1)}(\mu_x l_x) \\ +e^{(\ln v)(x+1)}(\mu_{x+1} l_{x+1})\end{matrix}\right\} \\ +2(\ln v)^2 e^{(\ln v)(x+1)}\left(\frac{d}{dx}d_x\right) + 3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2}d_x\right) \\ +e^{(\ln v)(x+1)}\frac{d^3}{dx^3}d_x\end{matrix}\right\}\right] \tag{91}$$

This is the continuous whole life insurance for a life aged x

Furthermore, from the results obtained in equation (86), we can value an increasing whole life insurance as the aggregate of the deferred whole life insurance schemes with deferred periods $0,1,2,3,\ldots$ years and the sum is unity so that the death benefits in the rth year becomes $(r+1)$. This reasoning applies irrespective of whether the death benefits is payable at the moment of death or at the end of the $\frac{1}{K}th$ year of death or at the end of year of death.

$$\left(IA^{(K)}\right)_x = \begin{cases} A^{(K)1}_{x:\overline{1}|} & + \ {}_1p_x(1+i)^{-1}A^{(K)1}_{x+1:\overline{1}|} & + \ {}_2p_x(1+i)^{-2}A^{(K)1}_{x+2:\overline{1}|} & + \ \ldots. \\ & + \ {}_1p_x(1+i)^{-1}A^{(K)1}_{x+1:\overline{1}|} & + \ {}_2p_x(1+i)^{-2}A^{(K)1}_{x+2:\overline{1}|} & + \ \ldots \\ & & + \ {}_2p_x(1+i)^{-2}A^{(K)1}_{x+2:\overline{1}|} & + \ \ldots \\ & & & + \ \ldots \\ & & & + \ \ldots \end{cases} \tag{92}$$

Now consider each row, we have

$$\left(IA^{(K)}\right)_x = A_x^{(K)} + {}_1p_x(1+i)^{-1}\left(A^{(K)1}_{x+1:\overline{1}|} + {}_1p_{x+1}(1+i)^{-1}A^{(K)1}_{x+2:\overline{1}|} + {}_2p_{x+1}(1+i)^{-2}A^{(K)1}_{x+3:\overline{1}|} + \ldots\right)$$
$$+ {}_2p_x(1+i)^{-2}\left(A^{(K)1}_{x+2:\overline{1}|} + {}_1p_{x+2}(1+i)^{-1}A^{(K)1}_{x+3:\overline{1}|} + {}_2p_{x+2}(1+i)^{-2}A^{(K)1}_{x+4:\overline{1}|} + \ldots\right) + \ldots \tag{93}$$

$$\left(IA^{(K)}\right)_x = A_x^{(K)} + {}_1p_x(1+i)^{-1}A_{x+1}^{(K)} + {}_2p_x(1+i)^{-2}A_{x+2}^{(K)} \tag{89}$$

$$\left(IA^{(K)}\right)_x = \left\{ \begin{array}{l} A_x - \frac{K-1}{2K}\left[\frac{C_x}{D_x}\right] - \frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_x}\left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+1)} \times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right) \\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right) \end{array} \right\}\right] \\ -\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_x}\left\{ \begin{array}{l} (\ln v)^2 \left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+1)} \times d_x - e^{(\ln v)(x+1)}\left(\mu_x l_x\right) \\ +e^{(\ln v)(x+1)}\left(\mu_{x+1} l_{x+1}\right) \end{array} \right\} \\ +2(\ln v)^2 e^{(\ln v)(x+1)}\left(\frac{d}{dx} d_x\right) + 3(\ln v)e^{(\ln v)(x+1)}\left(\frac{d^2}{dx^2} d_x\right) \\ +e^{(\ln v)(x+1)} \frac{d^3}{dx^3} d_x \end{array} \right\}\right] \end{array} \right\}$$

$$+_1 p_x (1+i)^{-1} A_{x+1} - \frac{K-1}{2K}\left[\frac{C_{x+1}}{D_{x+1}}\right] - \frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_{x+1}}\left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+2)} \times d_{x+1} - e^{(\ln v)(x+2)}\left(\mu_{x+1} l_{x+1}\right) \\ +e^{(\ln v)(x+2)}\left(\mu_{x+2} l_{x+2}\right) \end{array} \right\}\right]$$

$$-\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_{x+1}}\left\{ \begin{array}{l} (\ln v)^2 \left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+2)} \times d_{x+1} - e^{(\ln v)(x+2)}\left(\mu_{x+1} l_{x+1}\right) \\ +e^{(\ln v)(x+2)}\left(\mu_{x+2} l_{x+2}\right) \end{array} \right\} \\ +2(\ln v)^2 e^{(\ln v)(x+2)}\left(\frac{d}{dx} d_{x+1}\right) + 3(\ln v)e^{(\ln v)(x+2)}\left(\frac{d^2}{dx^2} d_{x+1}\right) \\ +e^{(\ln v)(x+2)} \frac{d^3}{dx^3} d_{x+1} \end{array} \right\}\right]$$

$$+_2 p_x (1+i)^{-2} \left\{ \begin{array}{l} A_{x+2} - \frac{K-1}{2K}\left[\frac{C_{x+2}}{D_{x+2}}\right] - \frac{\left(K^2-1\right)}{12K^2}\left[\frac{1}{D_{x+2}}\left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+3)} \times d_{x+2} - e^{(\ln v)(x+3)}\left(\mu_{x+2} l_{x+2}\right) \\ +e^{(\ln v)(x+3)}\left(\mu_{x+3} l_{x+3}\right) \end{array} \right\}\right] \\ -\frac{\left(K^4-1\right)}{720K^4}\left[\frac{1}{D_{x+2}}\left\{ \begin{array}{l} (\ln v)^2 \left\{ \begin{array}{l} (\ln v)e^{(\ln v)(x+3)} \times d_{x+2} - e^{(\ln v)(x+3)}\left(\mu_{x+2} l_{x+2}\right) \\ +e^{(\ln v)(x+3)}\left(\mu_{x+3} l_{x+3}\right) \end{array} \right\} \\ +2(\ln v)^2 e^{(\ln v)(x+3)}\left(\frac{d}{dx} d_{x+2}\right) + 3(\ln v)e^{(\ln v)(x+3)}\left(\frac{d^2}{dx^2} d_{x+2}\right) \\ +e^{(\ln v)(x+3)} \frac{d^3}{dx^3} d_{x+2} \end{array} \right\}\right] \end{array} \right\}$$

Discussion

The derived models generalize and unify existing actuarial formulations by incorporating an analytically rigorous estimation technique which extends beyond the traditional commutation functions approaches. However, the closed-form expression derived captures payment frequencies greater than one such as semi-annual, quarterly and monthly, resolving the problems of estimating act present values under non-annual payment assumptions. The Euler-Maclaurin expansion offers a smooth estimation to the sum of discounted probabilities over finely divided intervals, ensuring higher accuracy compared to the standard actuarial linear interpolation methods (uniform distribution of death assumption). The continuous whole life model derived through the Euler-Maclaurin series accommodates the limiting behavior of frequent payments as the interval tends to infinity. The resulting expression provides a tractable actuarial model, closely aligned to the analytical underpinnings of continuous-time life insurance mathematics and offers a basis for analytic comparison against more discretized methods.

The model for increasing whole life insurance, where the benefit due increases linearly over time, is particularly worrisome due to its analytical intractability. The application of the Euler-Maclaurin series enhances a tractable estimation to what would otherwise involve recursive methods. The derived expressions provides both analytical insight for practical opportunities application in product pricing and reserve estimation in life insurance products developments with benefit innovation characteristics.

Although market data is unavailable for direct numerical validation, the theoretical correctness of the results is supported by their derivation from the first principles and their internal consistency when compared with linear interpolation. For instance, when simplified under certain constraints (e.g. constant force of mortality or annual payments), the new models reduce to classical results, thereby reinforcing their validity.

The basic contribution of this paper lies in the derivation of these expressions and in demonstrating that the Euler-Maclaurin series, traditionally applied in numerical and analytical mathematics, can be efficiently deployed to the domain of actuarial mathematics for deriving advanced life insurance models. This analytical innovation widens the theoretical depth available to actuaries and lays the foundation for further research in estimating life contingency functions under complex payment and benefit structures.

Implications and Adequacy of the Derived Expressions

The implication of these results lies in their ability to enhance both the precision and computational efficiency of life insurance valuation. The multi-payment frequency model accounts for more realistic premium payment frameworks (monthly, quarterly or weekly), which are common in practice. Traditional actuarial models usually depend on estimations such as uniform distribution of deaths assumptions which assume annualized payments, leading to discrepancies. The closed-form expressions derived here eliminates the need for such estimations and aligns the model more closely with actual insurance contracts.

The continuous life insurance model provides exact limiting conditions of the discrete formulations, accommodating the behavior of infinitesimally small payment intervals. This is especially important in theoretical analyses, reserve computations under Solvency II or IFRS 17 frameworks, and in high-precision pricing conditions. The increasing whole life insurance model enables valuation of products where benefits increase linearly over time, which are commonly applied to hedge inflation. In the past, such products usually employed iterative numerical methods. The availability of a closed-form expression markedly ease-out implementation and sensitivity analysis.

Collectively, these tools provide enough and highly accurate technique for estimating the actuarial present value for different whole life insurance products, especially in theoretical or high-precision computational frameworks. By eliminating the need for summation over life contingency functions or numeric integration, they enhance the speed and reproducibility of actuarial computations.

Evidence Supporting the Adequacy of the Derivations

Although empirical validation using real-world mortality data and interest rate data were not conducted due to data unavailability, different lines of theoretical evidence support the validity and adequacy of the models. The analytical rigour involved in the derivations were based on the Euler-Maclaurin formula, which is a well-established technique of estimating sums by integrals with quantifiable error bounds. This lends analytical robustness to the expressions. In particular cases, such as annual payments, constant force of mortality or zero benefit escalation, the derived models reduce exactly to well-known classical actuarial expressions. This consistency strongly validates the generalizations proposed. The Euler-Maclaurin series provides known error bounds under certain smoothness conditions of the mortality laws and interest functions. Consequently, the models maintain a high level of accuracy even when data were not explicitly available, provided that assumptions on smoothness hold. Symbolic and numerical comparisons under hypothetical or constructed mortality laws will prove that the new models behave consistently with established theoretical expectations (e.g., actuarial present values increase with age and benefit size, and decrease with higher discount rates)

Conclusion

Life insurance estimations are essential based on two major reasons. The first reason evolves from the gap between the numerical estimations and analytical analysis. While numerical analysis sheds light on specified mortality scenarios, analytical techniques consider important properties and behaviour in general cases. This includes the asymptotic behaviour of mortality functions as the specified period of payment of benefits become large or infinitesimally small. The second reason concerns the challenge of implementing the approximation schemes. This paper therefore contributes in both directions as the results evince good understanding of such estimation procedures. Consequently, we investigate the effect of the Bernoulli power series on the behaviour of whole life insurance function in the long run. This method is important because we can generate a closed form expression which serves as a reference point in a more complex mortality scenario. The valuation of a life insurance policy still in force at any time $s;\ \ 0<s<\infty$ is essential to assess the solvency of the business. Since life insurance policy is essentially a long-term contract where the insurer accepts risk from the insured by receiving premiums and paying benefit when the contingent future event happens, we need to predict future events based on estimation. Therefore, some assumptions have to be made in respect of the variables of interest defined as the actuarial basis because life insurance policies depend on death or survival of the insured life in line with the economic and financial environment as premiums have to be invested to pay future benefits and on any other variables considered in the contract. However, with the emerging and sophistication of financial market in connection to the securitization of life insurance risk as an option in downplaying the traditional exchange of risk through reinsurance contracts, it becomes necessary to employ finance principles for the computation of life insurance premiums. Future work may include numerical validation of the models once suitable mortality and financial data become available, as well as extensions to incorporate stochastic interest rates or mortality improvements. Nevertheless, the results achieved here represent a significant theoretical advancement and contribute novel insights to the body of actuarial literature.

Recommendation

Insurance business is a risky business in terms of benefits paid out. In order to protect the life insurers from one-off pay out, the above model can be employed to compute the insured's benefits as agreed to, in the policy conditions.

References

Anggraeni A.S., Rahmadani S., Utama R.C., & Handayani V.A. (2023) Calculation of Indonesian pension funds using group self anuitization method and makeham mortality law. Jurnal Ilmu Keuangan dan Perbankan, Vol. 12, No. 2, pp. 269-280.

Bowers, L. N., Gerber, H. U., Hickman, C. J., Jones, D. A., Nesbitt, C.J., (1997) *Actuarial mathematics*. Society of Actuaries.

Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1985) A theory of the term structure of interest rates. Econometrica Vol. 53, pp. 385-408.

Christiansen, M. C. (2008) A sensitivity analysis concept for life insurance with respect to a valuation basis of infinite dimension. Insurance: Mathematics and Economics Vol. 42, pp. 680-690.

Christiansen, M. C. (2010) Biometric worst-case scenarios for multi-state life insurance policies. Insurance: Mathematics and Economics, Vol. 47, No. 2, pp. 190-197.

Dickson, D.C.M., Hardy, M.R. and Waters H.R. (2013) *Actuarial mathematics for life contingent risk*. 2nd edition. Cambridge University Press.

Hacaritz O, Kleinow T, Macdonald A.S. (2024) On technical bases and Surplus in life insurance. Scandinavian Actuarial Journal, pp. 1-29. https://doi.org./10.1080/03461238.2024.2363183

Hoem, J. M. (1969) Markov chain in life insurance. Blaetter Deutch. Gesellschaft Vers.math, Vol.9, pp. 91-107.

Hoem, J. M. (1988) The versatility of the Markov chain as a tool in the mathematics of life insurance. Transactions of the 23rd International Congress of Actuaries R, pp. 171-202.

Linnemann, P. (1993) On the application of Thiele's differential equation in life insurance. Insurance: Mathematics and Economics Vol.13, pp. 63-74.

Neil, A. (1979) *Life contingencies*, *The Institute of Actuaries and Faculty of Actuaries*, special ed, London.

Norberg, R. (1992b) Differential equation for moments of present values in life insurance. Insurance: Mathematics and Economics Vol. 17, pp. 171-180.

Ramlau-Hansen, H. (1990) Thiele's differential equation as a tool in product development in life insurance. Scandinavian Actuarial Journal, pp. 97-104.

Steffensen, M. (2000). A no arbitrage approach to Thiele's differential equation. Insurance: Mathematics and Economics Vol. 27, No. 2, pp. 201-214.

Sundt, B and Teugels, J. (2004) Encyclopedia of Actuarial Science, Wiley.